Isotropic nonarchimedean S-arithmetic groups are not left orderable

Groupes S-arithmétiques non-archimédiens isotropes ne sont pas ordonnés à gauche

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Abstract

If $\mathcal{O}$ is either $\mathbb{Z}[\sqrt{r}]$ or $\mathbb{Z}[1/r]$, where $r > 1$ is any square-free natural number, we show that no finite-index subgroup of $\mathrm{SL}(2, \mathcal{O})$ is left orderable. (Equivalently, these subgroups have no nontrivial orientation-preserving actions on the real line.) This implies that if G is an isotropic F-simple algebraic group over an algebraic number field F, then no nonarchimedean S-arithmetic subgroup of G is left orderable. Our proofs are based on the fact, proved by B. Liehl, that every element of $\mathrm{SL}(2, \mathcal{O})$ is a product of a bounded number of elementary matrices.

Résumé

Si $\mathcal{O}$ est soit $\mathbb{Z}[\sqrt{r}]$ ou soit $\mathbb{Z}[1/r]$, où $r > 1$ est un entier positif sans carré, nous prouvons qu'aucun sous-groupe d'indice fini de $\mathrm{SL}(2, \mathcal{O})$ n'est ordonné à gauche. (En d'autres mots, les sous-groupes d'indice fini de $\mathrm{SL}(2, \mathcal{O})$ ne possèdent pas d'action non triviale sur la droite respectant l'orientation.) Cela implique que si G est un groupe algébrique F-simple isotrope, défini sur un corps de nombres F, alors aucun sous-groupe S-arithmétique non-archimédien de G n'est ordonné à gauche. La démonstration est fondée sur le fait, due à B. Liehl, que chaque élément de $\mathrm{SL}(2, \mathcal{O})$ est le produit d'un nombre borné de matrices élémentaires.

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1. Introduction

It is known [8] that finite-index subgroups of $\mathrm{SL}(3,\mathbb{Z})$ or $\mathrm{Sp}(4,\mathbb{Z})$ are not left orderable. (That is, there does not exist a total order $\prec$ on any finite-index subgroup, such that $ab \prec ac$ whenever $b \prec c$.) More generally, if G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\, G \geq 2$, then no finite-index subgroup of $G_{\mathbb{Z}}$ is left orderable. It has been conjectured that the restriction on $\mathbb{Q}$-rank can be replaced with the same restriction on $\mathbb{R}$-rank, which is a much weaker hypothesis:

Conjecture 1 *If G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}\text{-rank}\, G \geq 2$, then no finite-index subgroup Γ of $G_{\mathbb{Z}}$ is left orderable.*

It is natural to propose an analogous conjecture that replaces $\mathbb{Z}$ with a ring of S-integers, and has no restriction on the $\mathbb{R}$-rank:

Conjecture 2 *If G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, and $\{p_1, \ldots, p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup Γ of $G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]}$ is left orderable.*

We prove Conjecture 2 under the additional assumption that $\mathbb{Q}\text{-rank}\, G \geq 1$:

Theorem 3 *If G is a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group, with $\mathbb{Q}\text{-rank}\, G \geq 1$, and $\{p_1, \ldots, p_n\}$ is any nonempty set of prime numbers, then no finite-index subgroup Γ of $G_{\mathbb{Z}[1/p_1,\ldots,1/p_n]}$ is left orderable.*

More generally, if G is an F-simple algebraic group over an algebraic number field F, with $F\text{-rank}\, G \geq 1$, then no nonarchimedean S-arithmetic subgroup Γ of G is left orderable.

We also prove some cases of Conjecture 1 (with $\mathbb{Q}\text{-rank}\, G = 1$). For example, we consider $\mathbb{Q}$-forms of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$:

Theorem 4 *If $r > 1$ is any square-free natural number, then no finite-index subgroup Γ of $\mathrm{SL}\big(2,\mathbb{Z}[\sqrt{r}]\big)$ is left orderable.*

In geometric terms, the theorems can be restated as the nonexistence of orientation-preserving actions on the line:

Corollary 5 *If Γ is as described in Theorem 3 or Theorem 4, then there does not exist any nontrivial homomorphism $\varphi\colon \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$.*

Combining this corollary with an important theorem of É. Ghys [3] yields the conclusion that every orientation-preserving action of Γ on the circle S^1 is of an obvious type; any such action is either virtually trivial or semiconjugate to an action by linear-fractional transformations, obtained from a composition $\Gamma \to \mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Homeo}^+(S^1)$. See [4] for a discussion of the general topic of group actions on the circle.

It has recently been proved that certain individual arithmetic groups are not left orderable (see, e.g., [2]), but our results apparently provide the first new examples in more than ten years of arithmetic groups that have no left-orderable subgroups of finite index. They are also the only known such examples that have $\mathbb{Q}\text{-rank}\, 1$.

The theorems are obtained by reducing to the fact, proved by B. Liehl [5], that if $\mathcal{O} = \mathbb{Z}[1/(p_1 \ldots p_n)]$ or $\mathcal{O} = \mathbb{Z}[\sqrt{r}]$, then $\mathrm{SL}(2,\mathcal{O})$ has bounded generation by unipotent elements. (That is, the fact that $\mathrm{SL}(2,\mathcal{O})$ is the product of finitely many of its unipotent subgroups. For the general case of Theorem 3, we also note that Γ contains a finite-index subgroup of $\mathrm{SL}\big(2,\mathbb{Z}[1/p]\big)$, for some prime p.) We are able to prove the same reduction for certain other groups:

Theorem 6 *Suppose Γ is a finite-index subgroup of either*

(i) $\mathrm{SL}\big(2,\mathbb{Z}[1/r]\big)$*, for some natural number $r > 1$, or*

(ii) $\mathrm{SL}(2,\mathcal{O})$*, where $\mathcal{O}$ is the ring of integers of a number field F, and F is neither $\mathbb{Q}$ nor an imaginary quadratic extension of $\mathbb{Q}$, or*

(iii) an arithmetic subgroup of a quasi-split $\mathbb{Q}$-form of the $\mathbb{R}$-algebraic group $\mathrm{SL}(3,\mathbb{R})$.

If $\varphi\colon \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is any homomorphism, and U is any unipotent subgroup of Γ, then every $\varphi(U)$-orbit on $\mathbb{R}$ is bounded.

Corollary 7 *Suppose*

– Γ *is as described in Thm. 6, and*
– Γ *is commensurable to a group that has bounded generation by unipotent elements.*
Then every homomorphism $\varphi\colon \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ *is trivial. Therefore,* Γ *is not left orderable.*

Assuming a certain generalized Riemann Hypothesis, G. Cooke and P. J. Weinberger [1] proved that the groups described in part (ii) of Thm. 6 do have bounded generation by unipotent elements. Thus, if this generalized Riemann Hypothesis holds, then finite-index subgroups of these groups are not left orderable. See [5] for relevant results on bounded generation that do not rely on any unproved hypotheses, and see [6] for a recent discussion of bounded generation.

2. Proof of Theorem 6(i)

Notation 8 *For convenience, let*

$$\overline{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \qquad \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \qquad \hat{s} = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}$$

for $u, v \in \mathbb{Z}[1/r]$ *and* $s \in \{\, r^n \mid n \in \mathbb{Z} \,\}$.

Suppose some $\varphi(U)$-orbit on $\mathbb{R}$ is not bounded above. (This will lead to a contradiction.) Let us assume U is a maximal unipotent subgroup of Γ.

Let V be a subgroup of Γ that is conjugate to U, but is not commensurable to U. Then $V_{\mathbb{Q}} \neq U_{\mathbb{Q}}$. Because $\mathbb{Q}\text{-rank}\,\mathrm{SL}(2,\mathbb{Q}) = 1$, this implies that $V_{\mathbb{Q}}$ is opposite to $U_{\mathbb{Q}}$. Therefore, after replacing U and V by a conjugate under $\mathrm{SL}(2,\mathbb{Q})$, we may assume

$$U = \{\, \overline{u} \mid u \in \mathbb{Z}[1/r] \,\} \cap \Gamma \qquad \text{and} \qquad V = \{\, \underline{v} \mid v \in \mathbb{Z}[1/r] \,\} \cap \Gamma.$$

Because V is conjugate to U, we know that some $\varphi(V)$-orbit is not bounded above. Let

$$x_U = \sup\{\, x \in \mathbb{R} \mid \text{the } \varphi(U)\text{-orbit of } x \text{ is bounded above} \,\} < \infty$$

and

$$x_V = \sup\{\, x \in \mathbb{R} \mid \text{the } \varphi(V)\text{-orbit of } x \text{ is bounded above} \,\} < \infty.$$

Assume, without loss of generality, that $x_U \geq x_V$.

Fix some $s = r^n > 1$, such that $\hat{s} \in \Gamma$, and let $B = \langle \hat{s} \rangle U$. Because $\langle \hat{s} \rangle$ normalizes U, this is a subgroup of Γ. Note that $\varphi(B)$ fixes x_U, so it acts on the interval (x_U, ∞). Since $\varphi(B)$ is nonabelian, it is well known (see, e.g., [4, Thm. 6.10]) that some nontrivial element of $\varphi(B)$ must fix some point of (x_U, ∞). In fact, it is not difficult to see that each element of $\varphi(B) \smallsetminus \varphi(U)$ fixes some point of (x_U, ∞). In particular, $\varphi(\hat{s})$ fixes some point x of (x_U, ∞).

The left-ordering of any additive subgroup of $\mathbb{Q}$ is unique (up to a sign), so we may assume that

$$\varphi(\overline{u_1})x < \varphi(\overline{u_2})x \Leftrightarrow u_1 < u_2 \qquad \text{and} \qquad \varphi(\underline{v_1})x < \varphi(\underline{v_2})x \Leftrightarrow v_1 < v_2.$$

The $\varphi(U)$-orbit of x is not bounded above (because $x > x_U$), so we may fix some $u_0, v_0 > 0$, such that

$$\varphi(\underline{v_0})x < \varphi(\overline{u_0})x.$$

For any $\underline{v} \in V$, there is some $k \in \mathbb{Z}^+$, such that $v < s^{2k} v_0$. Then, because $\varphi(\hat{s})$ fixes x and $s^{-2k} < 1$, we have

$$\begin{aligned} \varphi(\underline{v})x &< \varphi(\underline{s^{2k} v_0})x = \varphi(\hat{s}^{-k} \underline{v_0} \hat{s}^k)x = \varphi(\hat{s}^{-k})\varphi(\underline{v_0})x \\ &< \varphi(\hat{s}^{-k})\varphi(\overline{u_0})x = \varphi(\hat{s}^{-k} \overline{u_0} \hat{s}^k)x = \varphi(\overline{s^{-2k} u_0})x < \varphi(\overline{u_0})x. \end{aligned}$$

So the $\varphi(V)$-orbit of x is bounded above by $\varphi(\overline{u_0})x$. This contradicts the fact that $x > x_U \geq x_V$.

3. Other parts of Theorem 6

(ii) The above proof of Case (i) needs only minor modifications to be applied with a ring $\mathcal{O}$ of algebraic integers in the place of $\mathbb{Z}[1/r]$. (We choose $s = \omega^n$, where ω is a unit of infinite order in $\mathcal{O}$.) The one substantial difference between the two cases is that the left-ordering of the additive group of $\mathcal{O}$ is far from unique — there are infinitely many different orderings. Fortunately, we are interested only in left-orderings of $U = \{\, \overline{u} \mid u \in \mathcal{O} \,\} \cap \Gamma$ that arise from an unbounded $\varphi(U)$-orbit, and it turns out that any such left-ordering must be invariant under conjugation by $\hat{s}$. The left-ordering must, therefore, arise from a field embedding σ of F in $\mathbb{C}$ (such that $\sigma(s)$ is real whenever $\hat{s} \in \Gamma$), and there are only finitely many such embeddings. Hence, we may replace U and V with two conjugates of U whose left-orderings come from the same field embedding (and the same choice of sign).

(iii) A serious difficulty prevents us from applying the above proof to quasi-split $\mathbb{Q}$-forms of $\mathrm{SL}(3, \mathbb{R})$. Namely, the reason we were able to obtain a contradiction is that if $\overline{u_0}$ is upper triangular, $\underline{v}$ is lower triangular, $\hat{s}$ is diagonal, and $\lim_{k\to\infty} \hat{s}^{-k}\overline{u_0}\hat{s}^k = \infty$ under an ordering of Γ, then $\lim_{k\to\infty} \hat{s}^{-k}\underline{v}\hat{s}^k = e$. Unfortunately, the "opposition involution" of $\mathrm{SL}(3, \mathbb{R})$ causes the calculation to result in a different conclusion in case (iii): if $\hat{s}^{-k}\overline{u_0}\hat{s}^k$ tends to ∞, then $\hat{s}^{-k}\underline{v}\hat{s}^k$ also tends to ∞. Thus, the above simple argument does not immediately yield a contradiction.

Instead, we employ a lemma of M. S. Raghunathan [7, Lem. 1.7] that provides certain nontrivial relations in Γ. These relations involve elements of both U and V; they provide the crucial tension that leads to a contradiction.

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